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CAPTURE OF ELECTRONS BY POSITIVE IONS WHILE PASSING THROUGH GASES

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ABSTRACT. This work extends that of Brinkman and Kramers on the capture of electrons by positive ions while passing through gases. Detailed mathematical working is reported, and it is shown that contrary to the opinion of Brinkman and Kramers, the probability of capture of an electron by the α-particle in the $2p$-orbit from the H-atom becomes much larger than that for the capture in the $1s$-orbit when the velocity falls below $2(2\pi e^2)/h$. For small velocities, the ratio goes on increasing.

An α-particle passing through a gaseous medium as in a cloud chamber produces a track consisting of ions formed round electrons liberated from the surrounding gas. It was noticed by Henderson (1923) and Rutherford (1930) that towards the end of the track the phenomenon was more complex. They found that when the velocity had slowed down to a value comparable to $2c\alpha$ ($c\alpha$ = the velocity of the outer electron in the normal level of the H-atom in the molecules of the gas through which the α-ray passes) the α-particle might capture an electron and be converted to He^+, this might again lose its electron on collision with matter. The phenomenon of alternate loss and capture may occur a large number of times, but ultimately when the particles have sufficiently slowed down, most of them would permanently acquire an electron and be He^+. As He^+ further passes through the gas, it may capture an electron and become neutral helium. As it has a velocity of the order of 10^7 to 10^8 cm. sec.$^{-1}$ it may again lose an electron by collision with matter. This process of alternate loss and capture may continue for some more distance till the velocity slows down sufficiently and ultimately we get neutral helium atoms.

This phenomenon occurs at the last cm. of the path of the α-particle so that the experimental technique for its observation is rather difficult. The work of Rutherford and Henderson was continued on by Jacobsen (1930) who calculated the capture and loss cross-section from his own experimental works in which the motion of α-particles in air and hydrogen was studied; and he tried to compare his results with the theoretical conclusions of the authors mentioned below.

The theoretical study of this phenomenon was started by Fowler (1924), Thomas (1927), Oppenheimer (1928) and Brinkman and Kramers (1930). Fowler and Thomas used entirely classical conceptions and they need not be further considered. The work of Oppenheimer (1928) has been criticised by Brinkman and Kramers (1930) as one not free from objection. The latter authors have carried out by two alternative methods the calculations for finding

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out the capture cross-section. They assume that the atom of the gas through which the α-particles pass are hydrogen-like and the electrons move in $1s$-orbits, the capture also is assumed to take place in $1s$-orbits. The results of their calculations are compared with the experimental data of Rutherford and Henderson and of Jacobsen, and on certain assumptions, they are found to be in good agreement. Brinkman and Kramers (1930) have not, however, calculated the general case when the electrons may be moving in any kind of orbit in the atoms composing the gaseous medium and can be captured in any orbit by the α-particles. The restrictions are evidently for the purpose of simplifying the calculations. The laboratory medium mostly consists of nitrogen, oxygen and hydrogen molecules; though it may be possible to represent the motion of their outermost electrons by proper ψ-functions, the calculations with such ψ-functions may be extremely difficult, if not altogether impracticable to carry out. But if we stick to the approximation of Brinkman and Kramers as far as the traversed gaseous molecules are concerned, it is surely feasible to calculate the capture of electrons in orbits higher than $1s$ by α-particles. Brinkman and Kramers have not carried out this calculation because as they remarked correctly that such cross-sections are likely to be small compared with that of capture in $1s$ orbit. They have, however, given an expression for capture cross-section from $1s$-orbit to ns and *vice versa*.

The question of capture of electrons to higher p, d and f orbits by the α-particle, however, acquires a new importance in view of the recent suggestion (Saha, 1942) that helium lines occurring in the solar atmosphere may originate in this way: The problem of occurrence of helium lines in the sun has been for long a challenge to astro-physicists. It is well-known that none of the He-lines are found in the Fraunhofer absorption spectrum of the sun. This is as expected because at the temperature prevailing in the sun, He can exist only in normal state and as we know the absorption lines of normal helium are in the region $\lambda 584A$ to $\lambda 500A$, we cannot expect to observe them. The visible lines of He are all due to the excited states and the lowest of these states has $1s\,2s\ {}^3S_1$ an excitation potential of nearly 20.55 volts, which is not possible to have in an atmosphere having a temperature of 6000-7000°K. But when we turn to the spectrum of the chromosphere, we find that the visible lines of neutral helium are extremely strong, even the well-known line of ionised helium $\lambda 4686A$ is found to occur in it. This is rather an unexpected phenomenon, because this line has an excitation potential of nearly 75.25 volts, while the ordinary excitation in the chromosphere is 0 to 14 volts. There are certain other anomalous features in the occurrence of these lines. Evershed first noticed that the intensity of the He-lines appears to vanish near the limb and they are prominent only at some distance. This phenomenon has been more systematically studied by Perepelkin and Melnikov (1935) and Pannekoek and Minnaert (1928); the results obtained by the first mentioned authors on the variation of intensity of well-known D_3-line of helium is given in fig (1). It is seen that the line tends to vanish at the limb and attains its maximum intensity at a height of 2500 km. and beyond that it tends to vanish, though it can be traced up to 7500 km. The ionised line $\lambda 4686$ occurs in the lower chromosphere up to a height of 2000 km. only. These facts are rather in sharp contradiction to the ionisation theories as we move up in the solar atmosphere.

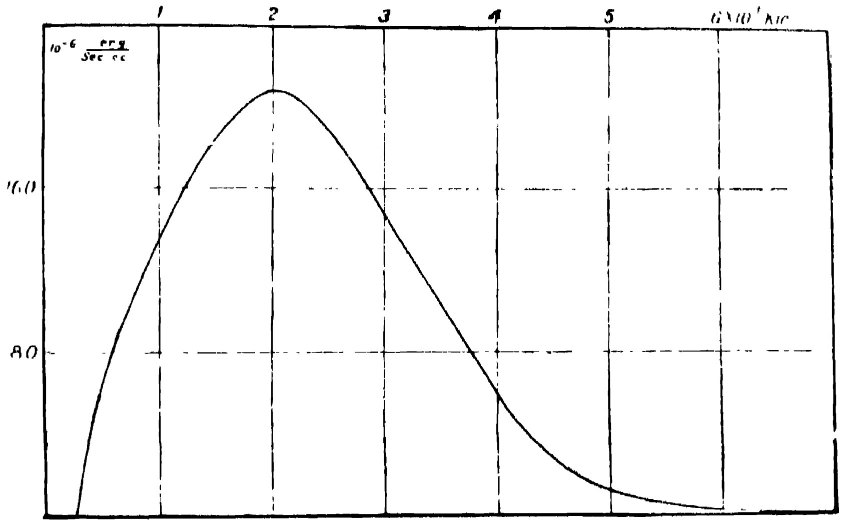

FIG. 1

Emission of 1cc. of Helium chromosphere in D_3-line in ergs per second and solid angle 4π

This phenomenon will probably receive a ready explanation if, as already suggested (Saha, 1942), it is assumed that the helium is not an ordinary constituent of the solar atmosphere; it is taken that due to some nuclear process, α-particles are being constantly generated throughout the solar body and some of them quite near the limb. We need not specify the particular nuclear reaction responsible for the generation of the α-particles as we can make our choice from a host of laboratory experiments. It is well-known that α-particles spontaneously emitted in radioactive disintegration have velocities of the order $6r\alpha$ to $10c\alpha$ (energy 4MV to 10MV).

It is clear that when such α-particles pass through the solar atmosphere, they will, during the first part of the motion, go on ionizing the solar gases (ionisation by collision), and losing energy; when they have sufficiently slowed down, they will begin to capture electrons in different orbits $1s$, $2p$, $3d$, ms, mp, and md. When the electron is captured in any orbit higher than $1s$, we have an excited He^+-atom which will emit a characteristic spectral line and revert back to a lower state. Most of these lines lie in the extreme ultra-violet, and the only one of He^+-line available for observation is $\lambda 4686$, $\nu = 4R\left[\frac{1}{3^2} - \frac{1}{4^2}\right]$. The capture takes place in any one of the $4f$, $4d$ or $4p$, $4s$ orbit and the line is emitted when the electron jumps back to any one of the $3d$, p, s orbits. All other lines of He^+ are outside the limit of experimental observation. This discussion brings out the necessity of calculating the cross-sections for the capture of electrons in orbits higher than $1s$. This has been attempted in the following sections. We have supposed that the solar gas through which the α-particles pass is entirely composed of hydrogen atoms. This very nearly represents the current idea according to which hydrogen forms more than 90% of the solar atmosphere.

When we turn to the next problem of occurrence of He-lines in the solar chromosphere the mathematical difficulties considerably increase. We cannot represent the field of He^{+} as even approximately coulombian and hence we must resort to very complicated calculations which have been used by Hylleras (1933, 1937) for finding out the He-terms and their transition probabilities. This has not yet been done, but if time permits it will form the subject of a discussion in another paper.

The procedure which we have followed in the calculation of capture of electron from $1s$- to $2p$-orbit is similar to the second method as adopted by Brinkman and Kramers in their parallel calculation of $1s$- to $1s$-capture. One must bear in mind that the method holds good only when (1) the gaseous molecules of the medium can be regarded as stationary during and after collision and when (2) the velocity of the α-particle is large compared to $c\alpha$.

Obviously these limitations put severe restrictions to the application of the results of the following calculation to solar phenomena. The first limitation means that the molecules or atoms must be very heavy compared to the α-particle; this is far from being the case with hydrogen atoms. The second one has been noted by all workers, but no alternative method has been put forward. It has been suggested that calculations of the Born approximation to second order may be carried out, but in view of past experience the suggestion does not appear very promising.

In spite of these limitations, we have proceeded with the calculations. In view of the fact that we have followed closely the method of Brinkman and Kramers, it is found useful to rewrite the essential steps of the above authors in the first part of this paper. That will be of help to appreciate the extension we have made of their method.

We suppose that the charged particle (α-particle here) having the mass M and the charge $Z'e$ moves past the atom K which has the mass Mq and the effective charge Ze (without the electron). The type of collision that we propose to study here is the following one: Initially we have an electron moving in the field of the atom, but after collision the same electron is captured by the α-particle in any one of its orbits. The reaction may be schematically represented as follows:

$$\begin{array}{ccccccc} K & + & He^{++} & = & K^{+} & + & He^{+} \\ (Mq+m,\ Ze) & & (M,\ 2e) & & (Mq.\ Ze) & & (M+m,\ 2e) \\ e & & \longrightarrow & & & & e \end{array}$$

To simplify calculation it is further assumed that Mq is so heavy that its nucleus remains at rest before and after collision.

Let K, α and e represent respectively the atom, the α-particle and the electron which in the beginning of the process belongs to K, and at the end is captured by the α-particle. We further choose the direction of motion of the α-particle as the axis of X, the perpendicular from the centre of the atom to the direction of

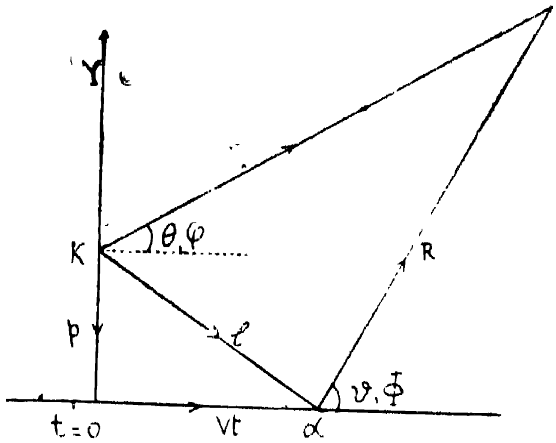

FIG. 2

motion as the Y-axis and Z is perpendicular to both. The quantities that will frequently occur are

p, collision parameter.
R, radius vector from α-particle to electron.
r, radius vector from the centre of the atom to electron.

Let us suppose that the α-particle is moving with velocity V, and in time $t = 0$, it is at the origin of the co-ordinate system. Then we have at any instant of time

$$\mathbf{R} = \mathbf{r} - \mathbf{p} - \mathbf{V}t \tag{1}$$

The total cross-section for such type of collision is

$$Q = \int_0^\infty 2\pi p \, |b|^2 dp \tag{2}$$

where b according to the perturbation theory is given by

$$\frac{h}{2\pi i}\frac{db}{dt} = \int \psi_i \frac{2e^2}{R} \psi_f \, d\tau \tag{3}$$

Now ψ_i is the wave function of the electron as attached to the atom in the initial state, while ψ_f is the wave function of the same as attached to the α-particle in the final state. It is easy to see that ψ_f consists of two parts:

$$\psi_f = \psi_{f,\,orb} \times \psi_{f,\,tr}$$

where $\psi_{f,\,orb}$ is the ordinary ψ-function due to the orbital motion of the captured electron round the α-particle and $\psi_{f,\,tr}$ is due to the translation motion of the electron which it shares because of its being attached to the moving α-particle.

$$\psi_{f,\,tr} = exp\left\{ -\frac{2\pi i}{h} \cdot \tfrac{1}{2} m V^2 t + \frac{2\pi i m}{h} (\mathbf{V}\mathbf{r}) \right\}$$

§ 1

For the case of $1s$ to $1s$-capture, we have the ψ-functions:—

$$\psi_{i,1s} = \frac{1}{\sqrt{\pi a_z^3}} exp\left\{-\frac{r}{a_z} - 2\pi i\nu_0 t\right\} \tag{4a}$$

$$\psi_{f,1s} = \frac{1}{\sqrt{\pi a_{no}^3}} exp\left\{-\frac{R}{a_{ne}} - 2\pi i\nu t + \frac{2\pi i}{h} m(\mathbf{V}.\mathbf{r}) - \frac{2\pi i}{h}\frac{mV^2 t}{2}\right\} \tag{4b}$$

where $h\nu_0$ = energy of the electron attached to the atom in the $1s$-orbit,

$h\nu$ = energy of the same attached to α-particle in the $1s$-orbit.

$a_z = a/z$, $a_{no} = a/z'$, where a is the Bohr-radius $= \frac{h^2}{4\pi^2 e^2 m}$, z = charge on atom and z' = charge on α-particle $= 2$; z' is retained for the sake of uniformity of notation.

Substitution of (4) in (3) gives us :

$$\frac{h}{2\pi i}\frac{db}{dt} = \frac{2e^2}{\pi(a_z a_{ne})^{3/2}} exp\{2\pi i\beta t\}.\int exp\left\{-\frac{r}{a_z} - 2\pi i(\sigma, \mathbf{r}) - \frac{R}{a_{ne}}\right\} d\tau_r . d\tau_R \tag{5}$$

where $\beta = \nu - \nu_0 + \frac{mV^2}{2h}$, and $\sigma = \frac{m\mathbf{V}}{h}$ (6)

To evaluate (5), use has been made of the following Fourier-integral

$$\frac{1}{R} exp\left\{-\frac{R}{a}\right\} = 4\pi a^2 \int \frac{exp\{2\pi i(\mathbf{R}.\mathbf{q})\}}{1 + 4\pi^2 a^2 q^2} \tag{7}$$

where $\mathbf{q}$ is any arbitrary vector having the dimension of $1/$Length.

We have

$$(\mathbf{q}.\mathbf{R}) = -q_x Vt - q_y p + (\mathbf{q}.\mathbf{r}), \tag{8}$$

then we may rewrite (5) as

$$\frac{h}{2\pi i}\frac{db}{dt} = \frac{8e^2}{(a_z a_{ne})^{3/2}} exp\left\{2\pi i V\left(\frac{\beta}{V} - q_x\right)t\right\}$$

$$\times \int \frac{exp\{-1/a_z + 2\pi i(\mathbf{q} - \sigma, \mathbf{r}) - 2\pi i q_y p\}}{(1/a_{ne}^2) + 4\pi^2 q^2} . r^2 dr \sin\theta\, d\theta d\phi d\mathbf{q} \tag{9}$$

Integration with respect to 't' gives us the well-known δ-function of Dirac (*vide*, Dirac, Quantum Mechanics (1935), p. 72),

$$\frac{h}{2\pi i} b = \frac{8e^2}{(a_z a_{ne})^{3/2}}\frac{1}{V}\int \frac{\delta\{\beta/V - q_x\} exp\{-1/a_z + 2\pi i(q - \sigma)r \cos\theta - 2\pi i p . q_y\}}{(1/a_{ne}^2) + 4\pi^2 q^2}$$

$$\times\, r^2 dr \sin\theta d\theta d\phi . d\mathbf{q} \tag{10}$$

The integration with respect to q_z follows from the properties of the δ-function

$$\frac{h}{2\pi i}b=\frac{8e^2}{(a_z^3 a_{\mathrm{He}}^3)^{1/2}}\frac{1}{V}\left\{\int\frac{exp\{-r/a_z+2\pi i(q-\sigma r)\cos\theta-2\pi i q_y p\}}{(1/a_{\mathrm{He}}^2)+4\pi^2q^2}\right.$$

$$\left.\times r^2 dr\sin\theta\, d\theta d\phi . dq_y dq_z\right\}_{q_x=\beta/V} \quad (11)$$

Integrating with respect to θ, ϕ, r, we have

$$\frac{h}{2\pi i}b=\frac{64\pi e^2}{(a_z^5 a_{\mathrm{He}}^2)^{1/2}}\frac{1}{V}\left\{\int\frac{exp\{-2\pi i q_y . p\}dq_y\, dq_z}{[(1/a_z^2)+4\pi^2(q-\sigma)^2]^2[(1/a_{\mathrm{He}}^2)+4\pi^2q^2]}\right\}_{q_x=\beta/V} \quad (12)$$

From the energy-principle, the two factors in the denominator are equal (*vide* appendix 1). Hence

$$\frac{h}{2\pi i}b=\frac{64\pi e^2}{(a_z^5 a_{\mathrm{He}}^3)^{1/2}}\frac{1}{V}\int\frac{exp\{-2\pi i q_y p\}\, dq_y\, dq_z}{[(1/a_{\mathrm{He}}^2)+4\pi^2\{(\beta^2/V^2)+q_y^2+q_z^2\}]^3} \quad (13)$$

$$=\frac{64\pi e^2}{(a_z^5 a_{\mathrm{He}}^3)^{1/2}}\frac{1}{V}\frac{3}{16}\int\frac{exp\{-iyp\}}{(g^2+y^2)^{5/2}} \quad (14)$$

where

$$y=2\pi q_y, \qquad g^2=\frac{1}{a_{\mathrm{He}}^2}+\frac{4\pi^2\beta^2}{V^2}$$

or

$$|b| = \frac{2^3.\pi e^2}{(a_z^5 a_{\mathrm{He}}^3)^{1/2}}\cdot\frac{1}{hV}\left(\frac{1}{a_{\mathrm{He}}^2}+\frac{4\pi^2\beta^2}{V^2}\right)^{-2}x^2K_2(x), \quad x=pg \quad (15)$$

where

$$K_\nu(x)=\tfrac{1}{2}\pi i\, exp\frac{\pi\nu i}{2}\, H_\nu^1(exp\{i\pi/2\}x),$$

H_ν^1 being the Hankel function of the first kind (Copson, 1933). Substituting this value of $|b|$ in (2), we have

$$Q=\frac{2^7.\pi^3 e^4}{(a_z^5 . a_{\mathrm{He}}^3)^{1/2}}\cdot\frac{1}{h^2V^2}\left(\frac{1}{a_{\mathrm{He}}^2}+\frac{4\pi^2\beta^2}{V^2}\right)^{-5}\int x^5|K_2(x)|^2 dx$$

$$=\frac{2^{12}\pi^3}{5(a_z^5 . a_{\mathrm{He}}^3)}\cdot\frac{e^4}{h^2V^2}\left(\frac{1}{a_{\mathrm{He}}^2}+\frac{4\pi^2\beta^2}{V^2}\right)^{-5} (\textit{vide}\text{ appendix 3}). \quad (16)$$

Replacing the value of β, and after some calculation, we have

$$Q=\frac{2^{12}\pi^3 e^4}{5(a_z^5 a_{\mathrm{He}}^3)}\frac{e^4}{h^2V^2}\left[\frac{1}{2a_{\mathrm{He}}^2}+\frac{1}{a_z^2}+\pi^2\sigma^2+\frac{\left(\frac{1}{2a_{\mathrm{He}}^2}-\frac{1}{a_z^2}\right)^2}{16\pi^2\sigma^2}\right]^{-5} \quad (17)$$

Putting $V=c\alpha s$ we obtain after some reduction

$$Q=\frac{2^{20}\pi a^2 z^5 z'^3}{5}s^8 . [\{s^2+(z+z')^2\}\{s^2+(z-z')^2\}]^{-5} \quad (18)$$

This is the formula given by Brinkman and Kramers for capture from $1s$ to $1s$-orbits .

§ 2. CAPTURE IN THE 2p-ORBIT

Let us now proceed with the calculation of the cross-section for the capture of the electron from $1s$ to $2p$-orbits. The ψ-function for the initial $1s$-orbit is the same as before. The ψ-function for the $2p$-orbit is now triple, corresponding to the magnetic quantum numbers $m=0, \pm 1$. We have

$$\psi_{f,2p}=\frac{1}{4\sqrt{\pi a_{\mathrm{ne}}^{2}}}\cdot\frac{R}{a_{\mathrm{ne}}}\,exp\left\{-\frac{R}{2a_{\mathrm{ne}}}-2\pi i\nu t-\frac{2\pi i}{h}\cdot\frac{mV^2t}{2}+\frac{2\pi i m}{h}(\mathbf{V}.\mathbf{r})\right\}\left\{\begin{matrix}\frac{\cos\vartheta}{\sqrt{2}}\\ \frac{\sin\vartheta\, exp\{\pm i\phi\}}{2}\end{matrix}\right\} \quad \ldots (19)$$

The upper one is for $m=0$, the lower for $m=\pm 1$.

Here ϑ, Φ denote the polar angles of the electron with reference to the α-particle as origin. But in shifting the origin to the nucleus K which, in our approximation, is at rest, the new angles θ and ϕ are connected with ϑ, and ϕ in the following way

$$R\cos\vartheta=(r\cos\theta-Vt),\quad R\sin\vartheta\, exp\{\pm i\Phi\}=(r\sin\theta\, exp\{\pm i\phi\}-p) \qquad (20)$$

The Fourier-integral here takes the form

$$\frac{1}{R}\,exp\{-R/2a_{\mathrm{ne}}\}=2\pi a_{\mathrm{ne}}^{2}.\int\frac{exp\{i\pi(\mathbf{q}.\mathbf{R})\}d\mathbf{q}}{1+4\pi^2a_{\mathrm{ne}}^{2}\,q^2} \qquad (21)$$

We obtain as before

$$\frac{h}{2\pi i}\frac{db}{dt}=\frac{e^2}{4\sqrt{a_z^3a_{\mathrm{ne}}^5}}\,exp\left\{2\pi i V\left(\frac{\beta}{V}-\frac{q_x}{2}\right)t\right\}.$$

$$\int\frac{exp\left\{-\frac{r}{a}+2\pi i\left(\frac{q_x}{2}-\sigma\right)r\cos\theta-\pi i p q_y\right\}}{\left\{\frac{1}{4a_{\mathrm{ne}}^{2}}+\pi^2(q_x^2+q_y^2+q_z^2)\right\}}\times$$

$$exp\{\pi i\, r\sin\theta(q_x.\cos\phi+q_z.\sin\phi)\}d\mathbf{r}.d\mathbf{q}\left\{\begin{matrix}(r\cos\theta-Vt)/\sqrt{2}\\ (r\sin\theta\, exp\{\pm i\phi\}-p)/2.\end{matrix}\right. \qquad (22)$$

Integrating with respect to t, with the aid of Dirac's delta-functions, we have

$$\frac{h}{2\pi i}\,b_{m=0}=\frac{e^2}{4\sqrt{2a_z^3a_{\mathrm{ne}}^5}}\,\frac{1}{V}\int\ldots d\mathbf{r}d\mathbf{q}F(r,q_x,q_y,q_z)$$

$$\left\{r\cos\theta.\delta\left(\frac{q_x}{2}-\frac{\beta}{V}\right)+\frac{1}{2\pi i}.\delta'\left(\frac{q_x}{2}-\frac{\beta}{V}\right)\right\} \qquad (23)$$

where

$F(r,\, q_xq_yq_z)$

$$=\frac{exp\left\{-\frac{r}{a_z}+2\pi i\left(\frac{q_x}{2}-\sigma\right).r\cos\theta-\pi i p q_y+\pi i r\sin\theta(q_y\cos\phi+q_z\sin\phi)\right\}}{\frac{1}{4a_{\mathrm{ne}}^{2}}+\pi^2(q_x^2+q_y^2+q_z^2)} \quad \ldots (24)$$

Integrating with respect to q_x, we have

$$\frac{h}{2\pi i}\cdot b_{m=0} = \frac{e^2}{2\sqrt{2a_z^3\cdot a_{He}^5}}\cdot\frac{1}{V}\int\ldots\Big[F(r, q_x/2=\beta/V, q_y, q_z)\, r\cos\theta\, d\mathbf{r}$$

$$-\frac{1}{2\pi i}\cdot F'(r, q_x/2=\beta/V, q_y, q_z)\, d\mathbf{r}\Big]dq_y\, dq_z \qquad (25)$$

Here F' denotes differentiation of F with respect to $q_x/2$, and the substitution of β/V for $q_x/2$. On simplification

$$b_{m=0} = \frac{4\pi^2 e^2\beta/V}{hV\sqrt{2a_z^3 a_{He}^5}}\cdot\int\frac{exp\left\{-\frac{r}{a_z}+2\pi i\left(\frac{\beta}{V}-\sigma\right)r\cos\theta-\pi i p q_y\, ic\cos(\phi-\chi)\right\}}{\left[\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right]}$$

$$\times r^2 dr\sin\theta\, d\theta\, d\phi\, dq_y\, dq_z \qquad (26)$$

where

$$c=\pi r\sin\theta\sqrt{q_y^2+q_z^2}=k\sin\theta.$$

$$\chi=\tan^{-1}(q_z/q_y).$$

Similarly

$$b_{m=\pm1} = \frac{i\pi e^2}{2\sqrt{a_z^3 a_{He}^5}}\cdot\frac{1}{hV}\int\ldots\frac{exp\left\{-\frac{r}{a}+2\pi i\left(\frac{\beta}{V}-\sigma\right)r\cos\theta-\pi i p q_y\right\}}{\left\{\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right\}}dq_y\, dq_z$$

$$\times exp\{ic\cos(\phi-\chi)\}.\{r\sin\theta.exp(\pm i\phi)-p\}\, r^2 dr\sin\theta\, d\theta\, d\phi \qquad (27)$$

The integration of (26 & 27) with respect to ϕ, gives us

$$b_{m=0} = \frac{8\pi^3 e^2\beta/V}{hV\sqrt{2a_z^3 a_{He}^5}}\cdot\int\frac{exp\left\{-\frac{r}{a}+2\pi i\left(\frac{\beta}{V}-\sigma\right)r\cos\theta-\pi i p q_y\right\}}{\left[\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right]^2}$$

$$J_0(k\sin\theta).r^2 dr\sin\theta\, d\theta\, dq_y\, dq_z \qquad (28)$$

and

$$b_{m=+1} = \frac{i\pi^2}{\sqrt{a_z^3 a_{He}^5}}\,\frac{e^2}{hV}\int\frac{exp\left\{-\frac{r}{a_z}-2\pi i\left(\frac{\beta}{V}-\sigma\right)r\cos\theta-\pi i p q_y\right\}}{\left\{\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right\}}$$

$$\times r^2 dr\sin\theta\, d\theta\, dq_y dq_z.\{r\sin\theta\, iJ_1(k\sin\theta)exp(+i\chi)-p\, J_0(k\sin\theta)\} \quad (29)$$

A similar integration with respect to θ gives us

$$b_{m=0} = \frac{16\pi^3 e^2\beta/V}{hV\sqrt{2a_z^3 a_{He}^5}}\cdot\int\frac{exp\left\{-\frac{r}{a_z}-\pi i p q_y\right\}\sin nr}{\left[\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right]^2}\cdot\frac{r}{n}dr.dq_y.dq_z \quad (30)$$

and

$$b_{m=1}=\frac{2i\pi^2e^2}{hV}\int\frac{exp\left\{-\frac{r}{a_t}\right\}\left(\frac{\sin nr}{nr}-\cos nr\right)\frac{im}{n^2}\,exp\{-\pi ipq_y\}}{\left\{\frac{1}{4a_{\rm He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right\}}\,r^2\,dr\,dq_y\,dq_z \tag{31}$$

where

$$n=\sqrt{l^2+m^2},\quad l=2\pi\left(\frac{\beta}{V}-\sigma\right),\quad m=\pi\sqrt{q_y^2+q_z^2}. \tag{32}$$

Integrating with respect to r, we have

$$b_{m=0}=\frac{32\pi^3e^2\beta/V}{hV\sqrt{2a_t^5a_{\rm He}^5}}$$

$$\times\int\frac{exp\{-\pi ipq_y\}dq_y\,dq_z}{\left\{\frac{1}{4a_{\rm He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right\}^2\left\{\frac{1}{a_t^2}+4\pi^2\left(\frac{\beta}{V}-\sigma\right)^2+\pi^2(q_y^2+q_z^2)\right\}^2} \tag{33}$$

$$b_{m=1}=\frac{4i\pi^2}{\sqrt{a_t^5a_{\rm He}^5}}\cdot\frac{e^2}{hV}\int\frac{exp\{-\pi ipq_y\}dq_ydq_z}{\left\{\frac{1}{4a_{\rm He}^2}+4\pi^2\frac{\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right\}}$$

$$\times\left\{\frac{4im\,exp\{i\chi\}}{\left[\frac{1}{a_t^2}+4\pi^2\left(\frac{\beta}{V}-\sigma\right)^2+\ldots\right]^3}-\frac{p}{\left[\frac{1}{a_t^2}+\ldots\right]}\right\} \tag{34}$$

It is easy to deduce from the law of conservation of energy that the two factors in the denominator are equal; *i.e.*,

$$\left\{\frac{1}{4a_{\rm He}^2}+\frac{4\pi^2\beta^2}{V^2}+\pi^2(q_y^2+q_z^2)\right\}=\left\{\frac{1}{a_t^2}+4\pi^2\left(\frac{\beta}{V}-\sigma\right)^2+\pi^2(q_y^2+q_z^2)\right\} \tag{35}$$

Making use of this relation, and integrating with respect to q_y and q_z, we have

$$\left.\begin{aligned} b_{m=0}&=\frac{2\pi}{3}\cdot\frac{2\pi\beta/V}{(2a_t^5\,a_{\rm He}^5)^{\frac{1}{2}}}\cdot\frac{e^2}{hV}g^{-6}\,x^3K_3(x)\\ b_{m=1}&=-\frac{i}{3}\cdot\frac{\pi}{(a_t^5\,a_{\rm He}^5)^{\frac{1}{2}}}\cdot\frac{e^2}{hV}\cdot g^{-5}x^3K_2(x)\end{aligned}\right\} \tag{36}$$

where

$$g^2=\left(\frac{1}{4a_{\rm He}^2}+4\pi^2\frac{\beta^2}{V^2}\right),\qquad x=pg.$$

Substituting the above values in expression (2) we obtain

$$\left.\begin{aligned} Q_{m=0}&=\frac{2^2.\pi^3}{3^2}\cdot\frac{(2\pi\beta/V)^2}{a_t^5\,a_{\rm He}^5}\cdot\frac{e^4}{h^2V^2}\cdot g^{-14}\int_0^\infty x^7dx\,\{K_3(x)\}^2\\ Q_{m=1}&=\frac{2\pi^3}{3^2a_t^5\,a_{\rm He}^5}\cdot\frac{e^4}{h^2V^2}\cdot g^{-12}\int_0^\infty x^7dx\{K_2(x)\}^2\end{aligned}\right\} \tag{37}$$

The general method to work out this type of integrals has been given in appendix 3. We have finally

$$\left.\begin{aligned} Q_{m=0} &= \frac{2^9}{7}\cdot\frac{\pi^3(2\pi\beta/V)^2}{a_z^5\, a_{He}^5}\cdot\frac{e^4}{h^2V^2}\left[\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}\right]^{-7} \\ Q_{m=1} &= \frac{2^7}{3\cdot 7}\cdot\frac{\pi^3}{a_z^5\, a_{He}^5}\cdot\frac{e^4}{h^2V^2}\left[\frac{1}{4a_{He}^2}+4\pi^2\frac{\beta^2}{V^2}\right]^{-6} \end{aligned}\right\} \tag{38}$$

Reducing still further, we have finally

$$Q_{m=0} = \frac{2^{19}}{7}\cdot\frac{\pi a^2 . z^5 z'^5 . s^{10} . [s^2+z^2-(z'^2/4)]^2}{[\{s^2+(z+z'/2)^2\}\{s^2+(z-z'/2)^2\}]^7} \tag{39}$$

$$Q_{m=1} = \frac{2^{17}}{3\cdot 7}\cdot\frac{\pi a^2 . z^5 z'^5 . s^{10}}{[\{s^2+(z+z'/2)^2\}\{s^2+(z-z'/2)^2\}]^6} \tag{40}$$

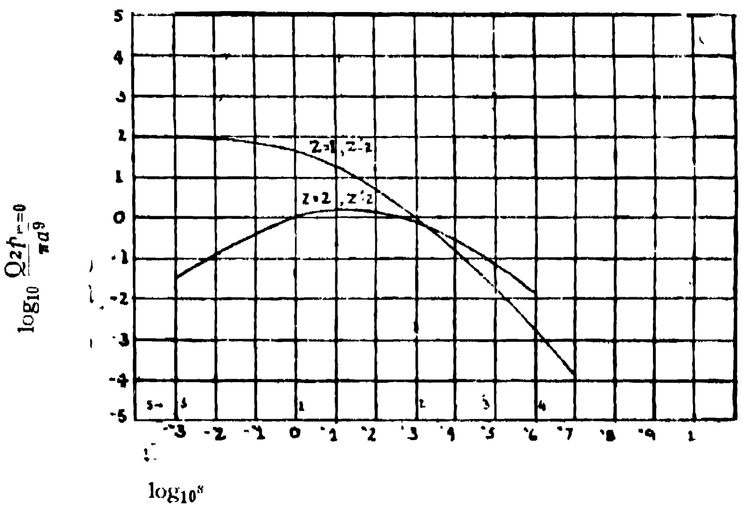

FIG. 3

DISCUSSION

Let us first compare the relative values of the different cross-section. We have for $z'=2$ and $z=1$.

$$\frac{Q_{2p_{m=\pm1}}}{Q_{2p_{m=0}}} = \frac{s^2+z^2}{6s^2} \simeq \frac{1}{6} \quad \text{when } s \text{ is large,}$$

$$\sim \frac{2}{3s^2} \quad \text{when } s \text{ is small.}$$

This shows that the probability of capture is much larger from 1s to $(2p)_{m=0}$ than from 1s to $(2p)_{m=\pm1}$, except when $s<<1$.

Let us next compare $Q_{2p_{m=0}}$ to Q_{1s}. We have from (39) and (40).

$$\frac{Q_{2p_{m=0}}}{Q_{1s}} = \frac{10}{7}\,\frac{(s^2+9)^5(s^2+1)^5}{s^8(s^2+4)^7} \quad \text{for the case } z'=2 \text{ and } z=1.$$

$Q_{2p_{m=0}}/Q_{1s}$ =	58.53	21.09	9.77	5.23	3.09	1.30	0.60	0.24
For s =	1.0	1.25	1.5	1.75	2.0	2.5	3.0	4.0

We thus obtain that for $z=1$ and $z'=2$, the capture cross-section $Q_{2p_{m=0}}$, though small compared to Q_{1s}, when $s>4$, is no longer so when s continues to fall.

At $V=2$, *i.e.*, when the α-particle has twice the velocity which the captured electron would have in its orbit, the ratio is nearly three times and at lower velocities it assumes a far higher value. Though there is some doubt whether these calculations can apply when s becomes small, it appears that the probability of capture in excited states increases rapidly with diminishing s, *i.e.*, towards the end of its path the charged particle would mostly be capturing the electron in the higher orbits. This is quite contrary to the view of Brinkman and Kramers that the capture probability in the higher orbits is negligible compared to that in the $1s$-orbit.

We have not yet been able to finish our calculation for capture in ns or higher nd or nf orbits. These will be taken up in a later paper.

APPENDIX 1

To prove

$$\frac{1}{a_z^2} + 4\pi^2\left\{\frac{\beta}{V} - \sigma\right\}^2 = \frac{1}{a_{n0}^2} + 4\pi^2\frac{\beta^2}{V^2}$$

we have

$$\beta = \nu - \nu_0 + \frac{mV^2}{2h}$$

where

$$\nu = -\frac{h}{8\pi^2 m a_{n0}^2}$$

$$\nu_0 = -\frac{h}{8\pi^2 m a_z^2}\ .$$

On substitution

$$\frac{\beta}{V} = \frac{1}{8\pi^2\sigma}\left(\frac{1}{a_z^2} - \frac{1}{a_{n0}^2}\right) + \frac{\sigma}{2}$$

where as usual

$$\sigma = \frac{mV}{h}$$

or

$$\frac{1}{a_z^2} - \frac{1}{a_{n0}^2} = \left(\frac{\beta}{V} - \frac{\sigma}{2}\right)8\pi^2\sigma = 4\pi^2\left\{\frac{\beta^2}{V^2} - \left(\frac{\beta}{V} - \sigma\right)^2\right\}$$

Hence follows the result.

APPENDIX 2

The integral

$$I_1 = \int_0^\pi e^{iu\cos\theta}\, J_0(k\sin\theta)\sin\theta d\theta$$

we have on putting the standard form of J_0,

$$I_1 = \Sigma \frac{(-1)^m (\frac{1}{2}k)^{2m}}{(m\,!)^2} \int_0^{\pi} e^{iB\cos\theta} (\sin\theta)^{2m+1} d\theta$$

$$= \frac{(-1)^m (\frac{1}{2}k)^{2m}}{(m\,!)^2} \frac{\Gamma(m+\frac{1}{2})\ \Gamma(\frac{1}{2})}{(\frac{1}{2}B)^{m+\frac{1}{2}}} J_{m+\frac{1}{2}}(B)$$

$$= \Sigma \frac{\Gamma(\frac{1}{2})\sqrt{2}}{m\,!} k^{2m} \left(\frac{d}{2BdB}\right)^m \left\{B^{-\frac{1}{2}} J_{\frac{1}{2}}(B)\right\}$$

$$= 2\Sigma \frac{1}{m\,!} \left(k^2 \frac{d}{2BdB}\right)^m \left\{\frac{\sin B}{B}\right\}$$

$$= 2\frac{\sin\sqrt{B^2+k^2}}{\sqrt{B^2+k^2}}.$$

The integral $\qquad I_3 = \int_0^{\pi} e^{iB\cos\theta}\ J_1(k\sin\theta)\sin^2\theta\ d\theta$

Let us take the integral $I_2 = \int_0^{\pi} e^{iB\cos\theta}\ J_0(k\sin\theta)\sin\theta\cos\theta\ d\theta$ and integrate the above by parts ; we have

$$I_2 = \frac{1}{k}\left[e^{iB\cos\theta}\sin\theta\ J_1(k\sin\theta)\right]_0^{\pi} + \frac{iB}{k}\int_0^{\pi} e^{iB\cos\theta}\ J_0(k\sin\theta)\sin^2\theta\ d\theta$$

The first term vanishes and it can be easily shown that

$$I_2 = -\frac{1}{i}\frac{dI_1}{dB},$$

so we have

$$I_3 = -\frac{k}{B}\frac{dI_1}{dB} = \frac{2k}{B^2+k^2}\left\{\frac{\sin\sqrt{B^2+k^2}}{\sqrt{B^2+k^2}} - \cos\sqrt{B^2+k^2}\right\}$$

APPENDIX 3

The evaluation of the integrals of the general type $\int_0^{\infty} x^m \{K_\nu(x)\}^2 dx$ is due to Dr. F. C. Auluck of the Delhi University to whom we express our thanks.

$$I_m = \int_0^{\infty} x^m dx \{K_\nu(x)\}^2 = \int_0^{\infty} x^m du \int_0^{\infty}\int_0^{\infty} e^{-x(\cosh t + \cosh t')} \cos\nu t \cos\nu t' dt dt'.$$

Further the integral on the right hand side is the coefficient of $(-1)^m (a^m/m\,!)$ in the integral

$$I = \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-x(a+\cosh t+\cosh t')} \cos\nu t \cos\nu t'\ dt dt' dx$$

$$= \int_0^\infty \int_0^\infty \frac{\cos \nu t \cos \nu t'}{a + \cosh t + \cosh t'} dt\, dt' \text{ (on integration with respect to } x)$$

$$= \tfrac{1}{2} \int_0^\infty \int_0^\infty \frac{\cosh \nu(t+t') + \cosh \nu(t-t')}{a + 2\cosh (t+t')/2 \cosh (t-t')/2}\, dt\, dt'$$

If we put

$$t + t' = 2x$$

$$t - t' = 2y$$

we get

$$I = \iint \frac{\cosh 2\nu x + \cosh 2\nu y}{a + 2\cosh x \cosh y} dx\, dy$$

$$= \int_{x=0}^\infty \int_{y=0}^x \frac{\cosh 2\nu x}{a + 2\cosh x \cosh y}\, dx dy + 2\int_{y=0}^\infty \int_{x=0}^\infty \frac{\cosh 2\nu y}{a + 2\cosh x \cosh y}\, dx dy$$

$$= 2\int_{x=0}^\infty \int_{y=0}^\infty \frac{\cosh 2\nu x\, dx dy}{a + 2\cosh x \cosh y}$$

$$= 2\int_{x=0}^\infty \frac{\cosh 2\nu x}{\sqrt{4\cosh^2 x - a^2}} \cos^{-1} \frac{2\cosh x + a\cosh y}{a + 2\cosh x \cosh y}\, dx \Bigg|_{y=0}^\infty$$

assuming $a < 2$.

$$= \int_0^\infty \frac{\cosh 2\nu x}{\sqrt{4\cosh^2 x - a^2}} \cos^{-1} \frac{a}{2\cosh x}\, dx$$

Expanding $\dfrac{1}{\sqrt{4\cosh^2 x - a^2}}$ and $\cos^{-1} \dfrac{a}{2\cosh x}$ in series

$$I = \int_0^\infty \frac{\cosh 2\nu x}{\cosh x} \left[\frac{\pi}{2} - \frac{a}{2\cosh x} + \frac{\pi}{2} \frac{1}{2!} \frac{a^2}{(2\cosh x)^2} - \frac{2^2}{3!} \frac{a^3}{(2\cosh x)^3} \right.$$

$$+ \frac{\pi}{2} \frac{1^2.3^2}{4!} \frac{a^4}{(2\cosh x)^4} - \frac{2^2.4^2}{5!} \frac{a^5}{(2\cosh x)^5}$$

$$\left. + \frac{\pi}{2} \frac{1^2.3^2.5^2}{6!} \frac{a^6}{(2\cosh x)^6} - \frac{2^2.4^2.6^2}{7!} \frac{a^7}{(2\cosh x)^7} \cdots \right] dx$$

Hence for $m \geq 4$

$$I_m = \int_0^\infty x^m dx \iint e^{-x(\cosh t + \cosh t')} \cosh \nu t \cosh \nu t'\, dt\, dt'$$

$$= \frac{2^2.4^2.6^2. \ldots .(m-1)^2}{2^m} \int_0^\infty \frac{\cosh 2\nu x}{\cosh^{m+1} x}\, dx \quad \text{if } m \text{ is odd}$$

$$= \frac{\pi}{2} \cdot \frac{1^2.3^2.5^2. \ldots .(m-1)^2}{2^m} \int_0^\infty \frac{\cosh 2\nu x}{\cosh^{m+1} x}\, dx \quad \text{if } m \text{ is even}$$

The cases that concern us are for $\nu = 2$ and $\nu = 3$. For $\nu = 2$, the expression under the integral may be written as

$$\int_0^\infty \frac{\cosh 4x}{\cosh^{m+1} x} = \frac{(m+1)(m+3)}{m(m-2)} \int_0^\infty \operatorname{sech}^{m-3} x \, dx$$

From which follows

$$^2I_4 = \frac{315}{256}\pi^2, \qquad {}^2I_5 = \frac{2^5}{5}$$

$$^2I_6 = \frac{3^3 \cdot 175}{2^{11}}\pi^2, \qquad {}^2I_7 = \frac{3 \cdot 2^6}{7}$$

Similarly for $\nu = 3$

$$^3I_7 = \frac{2^7 \cdot 3}{7}.$$

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